III. COMPARISON WITH THE EXACT RESULTS

Wu[5] has shown that for the first three terms in the exact ground-state energy of the spinless boson gas the results are shape-independent, and should depend only on the scattering length a. For the potential of Eq. (4), it is well known[6] that

$$a=r_0+(b-1)r_0[1-\tan\beta r_0(b-1)/\beta r_0(b-1)],$$

where

$$\beta=(mV_0)^{1/2}/\hbar. \tag{10}$$

Since the present calculation is, in effect, a correction to E of first order in V_0, the scattering length can be approximated by

$$a\approx r_0[1-mr_0^2V_0(b-1)^3/3\hbar^2]=r_0(1-\delta). \tag{11}$$

[5] T. T. Wu, Phys. Rev. **115**, 1390 (1959).

[6] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), pp. 111–112.

Replacing r_0 by a throughout Eq. (5), and keeping terms linear in δ, one obtains

$$E'=4\pi\rho r_0\frac{\hbar^2}{2m}\left[(1-\delta)+\frac{5\sqrt{\alpha}}{3\sqrt{3}}\left(1-\frac{5}{2}\delta\right)+\tfrac{3}{4}\alpha\ln\alpha(1-4\delta)\right]. \tag{12}$$

The comparison with Eq. (9) shows exact agreement for the first term (which is trivial), 4% agreement for the second term, and very poor agreement for the logarithmic term.

The present results would constitute a consistent first-order perturbation calculation if the unperturbed function [Eqs. (1) and (3)] were an exact eigenfunction of the hard-sphere problem. The errors obtained here for the various terms are qualitatively what was to be expected from the degree of agreement previously found for the hard-sphere case without attraction.[3]

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Regge Poles and Complex Singularities*

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We investigate the analyticity in complex angular momentum for the case in which complex singularities are present in double-spectral representations of scattering amplitudes. The simple example we consider is Σ-Σ scattering. We show that Regge continuation exists and has the same qualitative characteristics as for the case of no complex singularities. In particular, we find no direct connection between the presence of complex singularities and the existence of the branch cuts in angular momentum.

I. INTRODUCTION

SEVERAL authors[1] have investigated the Regge behavior of S-matrix elements with normal thresholds. In these investigations, it has been generally assumed that Regge poles are the only singularities in the complex angular momentum plane. Recently, however, it has been suggested[2,3] that branch points, with associated branch cuts, may also be present. The very slow approach of nuclear cross sections to their ultimate constant values has been attributed by Udgaonkar and Gell-Mann[3] to the presence of branch cuts. Since nuclei are composite objects with prominent anomalous thresholds, it is appealing to assume that the

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§ On deputation from Tata Institute of Fundamental Research, Bombay, India.

[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961) and **8**, 41 (1961); G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **121**, 1202 (1962); S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, *ibid.* **126**, 2204 (1962); R. Blankenbecler and M. L. Goldberger, *ibid.* **126**, 766 (1962); B. M. Udgaonkar, Phys. Rev. Letters **8**, 142 (1962); Virendra Singh, Phys. Rev. **129**, 1889 (1962).

[2] D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962).

[3] B. M. Udgaonkar and M. Gell-Mann, Phys. Rev. Letters **8**, 346 (1962).

branch cuts in the angular momentum plane are intimately associated with the presence of anomalous thresholds and complex singularities.

Unfortunately, an amplitude with complex singularities fails to obey the Mandelstam representation—which has been the starting point of all other investigations of Regge behavior. Assuming that the Mandelstam representation holds, Froissart has given a continuation of partial-wave scattering amplitudes to complex angular momenta, which is well behaved at infinity in the right-half-plane.[4] *A priori*, it is unclear that such a continuation can be given at all, or that any possible continuation would have the simple analytic properties so far assumed in the normal case, for amplitudes with complex singularities.

Recently, Fronsdal, Mahanthappa, and Norton[5] have given integral representations akin to the Mandelstam representation for certain simple scattering amplitudes with non-normal thresholds, using the Bergmann-Oka-Weil formula.[6] The virtue of these representations is that all integrals are taken along the real axes in the s, t planes, even though the amplitudes are singular on complex surfaces. Such formulas make it possible to extend the Regge formalism to a wider class of amplitudes. In this paper, we study the relatively simple example of Σ-Σ scattering with two particles (such as two Λ particles) as intermediate states. This should be sufficient in principle to elucidate any *direct* connection between Regge cuts and anomalous thresholds or complex singularities.

The Regge continuation of Σ-Σ amplitude turns out to be surprisingly well behaved. The partial waves for complex l have a behavior which is in almost every respect qualitatively the same as for partial waves of amplitudes with normal thresholds. The most important conclusion is that there seems to be no causal connection between complex singularities produced by two-particle intermediate states and Regge cuts. These are the following points of similarity between the normal and anomalous cases: (i) A Regge representation involving Q_l, the Legendre function of the second kind, can be defined. (ii) The partial-wave amplitudes $a(l,s)$ have only real singularities in s (although there are more branch points in the anomalous case). (iii) The spectral functions in the BOW formula are given by essentially the same unitarity formulas as in the normal case. (iv) The sharing theorem and factoring theorem holds (see Sec. IV). (v) Any proof of meromorphy given for a normal case can be extended to the anomalous case. It should be emphasized that all these points of similarity have been deduced using two-particle intermediate states. What happens when many-body thresholds are included is not discussed. Convenient BOW representations for a diagram with three-particle intermediate states, for example an eclipse diagram (Fig. 1), do not exist; such diagrams may well have cuts.

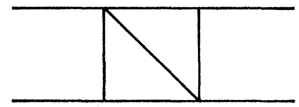

FIG. 1. An eclipse diagram.

In Sec. II, the BOW representation and unitarity formulas for spectral functions are developed; in Sec. III Regge continuation is given and its properties discussed, Sec. IV deals with the generalized unitarity and the Froissart identity; Sec. V contains the conclusion and mention of some recent work.

II. INTEGRAL REPRESENTATION

Consider the box diagram of Fig. 2. We define the variables $s=(p_{12}+p_{23})^2$, $t=(p_{12}+p_{14})^2$. The internal masses m_i may actually represent single-particle states, as indicated in the figure, or the line labeled m_i may represent a many-particle state of total invariant mass, m_i (in which case, the figure is to be interpreted as a Cutkosky reduced graph).

Tarski[7] has given the rules for deciding whether box diagrams have complex singularities or not. For simplicity, suppose each vertex is stable:

$$p_{i,i+1}^2 \leq (m_i+m_{i+1})^2.$$

Define

$$y_{ij}=1 \quad \text{for} \quad i=j$$
$$=-\frac{p_{ij}^2-m_i^2-m_j^2}{2m_i m_j} \quad \text{for} \quad i\neq j.$$

Then, with the condition of stability, one can define four real angles θ_i, $|\theta_i|<\pi$ by $\cos\theta_i=y_{i,i+1}$ $(y_{45}=y_{41})$. The condition for no complex singularities is $\sum \theta_i \leq 2\pi$. In the special case of "lowest box diagram for Σ-Σ scattering," where $p_{ij}^2=M^2$, $m_1=m_3=m$, and $m_2=m_4=\mu$; M, m, and μ being the masses of Σ, Λ, and π, because for physical values of Σ mass,[8] $M^2>m^2+\mu^2$ and all y_{ij} are negative, the condition for nonoccurrence of complex singularities is violated.

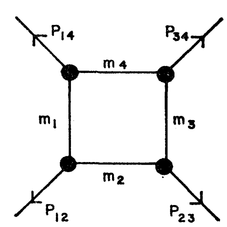

FIG. 2. General box diagram.

[4] M. Froissart, Nuovo Cimento **22**, 395 (1961).
[5] C. Frondsdal, K. Mahanthappa, and R. Norton, Phys. Rev. **127**, 1847 (1962); J. Math. Phys. **4**, 859 (1963), hereafter referred to as FMN.
[6] The references concerning Bergmann-Oka-Weil (BOW) formula are given in Ref. 5.
[7] J. Tarski, J. Math. Phys. **1**, 149 (1960).
[8] To avoid additional complex singularities except those from the box diagram, it is chosen that "physical Σ mass" be slightly less than its "experimentally observed mass."

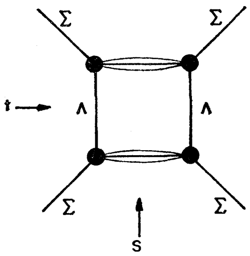

FIG. 3. Cutkosky graph related to the box diagram.

Therefore, in accordance with the principle of analytic continuation in M^2 (first used by Mandelstam[9]), we suppose M^2 to be lowered until $\sum \theta_i < 2\pi$. Then the box diagram (with $m_1=m_3=m$; $m_2=m_4=\mu$) admits a Mandelstam representation

$$A_B(s,t)\sim\int ds'dt'\frac{\rho_B(s',t')}{(s'-s)(t'-t)}, \tag{2.1}$$

where

$$\rho_B(s,t)=[\det|y_{ij}|]^{-1/2}.$$

More generally, let us interpret the box diagram of Fig. 2 as a Cutkosky graph (Fig. 3). The mass of Σ is to be taken sufficiently small that the amplitude $B(s,t)$ for $\Sigma\Sigma\rightarrow\Lambda\Lambda$, obeys the Mandelstam representation, as does $A(s,t)$ itself with a kernel now denoted by $\tilde{\rho}$. Then, from the work of Mandelstam,

$$\tilde{\rho}(s,t)=\frac{1}{32\pi^2q^3s^{1/2}}\int dt'dt''\,k^{-1/2}(s;t,t',t'')\times\{B_t^*(s,t')B_t(s,t'')+B_u^*(s,t')B_u(s,t'')\}. \tag{2.2}$$

Here, q is the c.m. momentum and subscripts indicate the absorptive part in the appropriate channel;

$$k^{1/2}(s;t,t',t'')/(32\pi^2q^3s^{1/2})$$

is just the kernel $\rho(s,t,[t',t''])$ for the box diagram of Fig. 3, evaluated for $m_1=m_3=m$; $m_4=(t'')^{1/2}$ and $m_2=(t')^{1/2}$. In the case of single-particle intermediate states, B_t and B_u are just δ functions. From this point on, to simplify the writing, we shall suppress the u channel. We may now construct $A(s,t)$ by integrating over $\tilde{\rho}$

$$A(s,t)\sim\int\frac{ds'dt'}{(s'-s)(t'-t)}dt_1dt_2\rho(s',t'[t_1,t_2])\times B_t^*(s',t_1)B_t(s',t_2), \tag{2.3}$$

where $\rho(s',t'[t_1,t_2])$ means the weight function for the box diagram with $m_1=m_3=m$; $m_2=(t_1)^{1/2}$ and $m_4=(t_2)^{1/2}$. So far we have assumed that M^2 is so small that no complex singularities exist for the minimum values of t_1 and t_2. As M^2 is increased[10] to the physical value of the Σ mass, complex singularities develop and Eq. (2.3) fails. But FMN have shown how to continue in M^2 the integral over ds', dt', at least for certain special cases, the most important of which is $t_1=t_2$. In this case, an integral representation involving only real paths of integration can be found by using the BOW formula. We refer the reader to FMN for details, and quote only the results for the box diagram setting $t_1=t_2=\mu^2$. Define $\bar{a}=(M^2-m^2-\mu^2)^2$. Then the box diagram of Fig. 2 is represented by

$$A_B\sim\int_{\mathrm{I}} ds'dt'\frac{\rho(s',t',[\mu^2,\mu^2])}{(s'-s)(t'-t)}+\int_{\mathrm{II}} ds'dt'\frac{(t'+t-8\mu^2)\rho(s',t',[\mu^2,\mu^2])}{(t'-t)[(s'-4m^2)(t'-4\mu^2)-(s-4m^2)(t-4\mu^2)]}+\left(\frac{s}{4m^2}\leftrightarrow\frac{t}{4m^2}\right)\cdots, \tag{2.4}$$

where the regions I and II are shown in Fig. 4. Region I is just the normal region in which $\rho>0$, for the case of no complex singularities, while region II is present only for the complex singularities.

For our choice of physical Σ mass, the box diagram with $t_1=\mu^2$, $t_2=\mu^2$ alone leads to complex singularities, and we have the normal Mandelstam representation for other Cutkosky diagrams with $t_1\geq 4\mu^2$, $t_2=\mu^2$ or $t_2\geq 4\mu^2$ and vice versa. We can, therefore, write

$$A(s,t)\sim\int_{\mathrm{I}}\frac{ds'dt'\rho(t's'),}{(s'-s)(t'-t)}+\left[\int_{\mathrm{II}}\frac{ds'dt'[t+t'-8\mu^2]\sigma(s',t')}{(t'-t)[(s'-4m^2)(t'-4\mu^2)-(s-4m^2)(t-4\mu^2)]}+\left(\frac{s}{4m^2}\leftrightarrow\frac{t}{4\mu^2}\right)\right], \tag{2.5}$$

where

$$\rho(s,t)=\int dt_1\int dt_2\,\rho(s,t;[t_1,t_2])B_t^*(s,t_1)B_t(s,t_2),$$

$$\sigma(s,t)=\int\int dt_1dt_2\,\rho(s,t;[t_1,t_2])B_t^{*(\mathrm{pole})}(s,t_1)B_t^{\mathrm{pole}}(s,t_2)$$
$$=\rho(s,t;[\mu^2,\mu^2])\left|\int B_t^{\mathrm{pole}}(s,t')dt'\right|^2,$$

and $B_t^{\mathrm{pole}}(s,t)$ is the contribution of pole in $B(s,t)$ at $t=\mu^2$ to $B_t(s,t)$; i.e., $B_t^{\mathrm{pole}}(s,t)\sim\delta(t-\mu^2)$.

So far, we have not included intermediate states other than the one with two Λ particles. The effect of their inclusion is simply to contribute additive terms to $\rho(s,t)$, in the integral over the region I, of the form

$$\sum_{m_1,m_3}\int dt_1\int dt_2\,\rho(s,t;[t_1,t_2];m_1,m_3)\times T_{m_1,m_3}^*(s,t_1)T_{m_1,m_1}(s,t_2),$$

where $T_{m_1,m_3}(s,t)$ is the absorptive part of the scat-

[9] S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).

[10] As M^2 is increased to the physical value of Σ mass, $B_t(s,t)$ and $B_t^*(s,t)$ are to be understood as including the anomalous supports of the integration region.

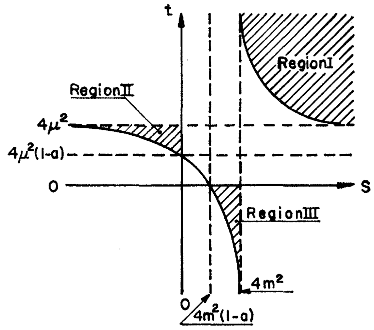

FIG. 4. Regions of integration in s, t plane for double spectral representation.

tering amplitude for the process $\Sigma\Sigma \to (m_1,m_2)$ and $\rho(s,t;[t_1,t_2];m_1,m_3)$ is the kernel for the box diagram of Fig. 2 with $t_1=m_2^2$ and $t_2=m_4^2$. We shall implicitly assume that these additive contributions to $\rho(s,t)$ have been incorporated in the spectral representation.

In the above development, we have been able to express $A(s,t)$ as an integral over real cuts only, even though $A(s,t)$ has complex singularities. The integral over the region II contains spectral functions, which are constructed by analytic continuation in mass, M, from the spectral functions, given by Eq. (2.2), which were obtained by using unitarity. The spectral functions are integrals over products of absorptive parts. This is important for the validity of the factoring theorem, as will be discussed in Sec. IV.

III. REGGE CONTINUATION

We shall now try to find a continuation into complex angular momentum l of the partial-wave amplitudes, which, (i) reduces to proper Regge continuation, given by Froissart,[4] when there are no complex singularities; (ii) is holomorphic to the right of some line $\mathrm{Re}l=N$ and vanishes at infinity for $\mathrm{Re}l>N$, and (iii) of course, agrees with physical partial-wave amplitudes for $l=0, 1, 2, \cdots$.

One sees, from expression (2.5) giving the representation, that the integral over region I, being Mandelstam type, admits the Regge continuation in terms of $Q_l(z)$ functions. It is the integral over the region II which is different and presents difficulties. Fortunately, there is a rather simple trick, when one realizes it, which makes it possible to write Regge continuation of partial-wave projection from this term also. We notice that we can rewrite the second term as a sum of two terms, the denominator of each of which is linear in t. Once we have this, it is easy to write down the Regge continuation in terms of $Q_l(z)$ functions again, though the argument of $Q_l(z)$ function is different from the one which occurs for the first integral over region I. Thus,

$$A(s,t)\sim\int_{\mathrm{I}}\frac{ds'dt'\rho(s',t')}{(s'-s)(t'-t)}+\left\{\int\!\!\int_{\mathrm{II}}\frac{ds'dt'}{s'-s}\left[\frac{2}{t'-t}-\frac{s+s'-8m^2}{(s-4m^2)(t'-4\mu^2)-(s-4m^2)(t-4\mu^2)}\right]\times\sigma(s',t')+\left(\frac{s}{4m^2}\leftrightarrow\frac{t}{4\mu^2}\right)\right\}. \quad (3.1)$$

Now we can explicitly write down the proper continuation

$$a(l,s)\sim\frac{1}{2q^2}\left\{\frac{1}{\pi^2}\int_{\mathrm{I}}\frac{ds'dt'\rho(s\,,t')}{(s'-s)}Q_l\left(1+\frac{t'}{2q^2}\right)+\frac{1}{\pi^2}\int_{\mathrm{II}}\frac{ds'dt'\sigma(s',t')}{(s'-s)}\left[2Q_l\left(1+\frac{t'}{2q^2}\right)-\frac{s+s'-8m^2}{(s-4m^2)}\right.\right.$$
$$\left.\left.\times Q_l\left(1+\frac{4\mu^2}{2q^2}+\frac{(s'-4m^2)(t'-4\mu^2)}{2q^2(s-4m^2)}\right)\right]+\frac{1}{\pi^2}\int_{\mathrm{III}}\frac{ds'dt'\sigma(s',t')}{(s'-s)}\frac{(s+s'-8m^2)}{(s-4m^2)}Q_l\left(1+\frac{4\mu^2}{2q^2}+\frac{(s'-4m^2)(t'-4\mu^2)}{2q^2(s-4m^2)}\right)\right\}$$
$$\equiv a_l^{(1)}(s)+a_l^{(2)}(s)+a_l^{(3)}(s)+a_l^{(4)}(s)\,, \quad (3.2)$$

where $s=4(q^2+M^2)$. The last integral in Eq. (3.2), i.e., $a_l^{(4)}(s)$, comes from the $(s/4m^2\leftrightarrow t/4\mu^2)$ part of the representation. In reducing that contribution to this form, we have used the fact that $\sigma(s,t)$ is essentially symmetrical under the operation $(s/4m^2\leftrightarrow t/4\mu^2)$, as can be seen from the explicit expression for $\sigma(s,t)$ given by

$$\sigma(s,t)\sim\left\{\left(\frac{s}{4m^2}\right)\left(\frac{t}{4\mu^2}\right)\left[\left(\frac{s}{4m^2}-1\right)\left(\frac{t}{4\mu^2}-1\right)-\left(\frac{M^2-m^2-\mu^2}{2m\mu}\right)^2\right]\right\}^{-1/2}\cdots. \quad (3.3)$$

Several important features emerge from the expression (3.2), which defines the continuation which is holomorphic in the region $\mathrm{Re}l>N$, N depending on the asymptotic behavior of $\rho(s,t)$, as in the normal case. When one continues $a_l(s)$ away from the region $\mathrm{Re}l>N$ into $\mathrm{Re}l<N$, assuming it to be possible, one will in general encounter Regge poles. In discussing the dispersion relations of Regge parameters,[11] i.e., the position and residue of the Regge poles, it is necessary to look at the singularities of $a_l(s)$ in the s plane, which we now proceed to do.

The singularities are located by finding those values

[11] Virendra Singh, Phys. Rev. **127**, 632 (1962).

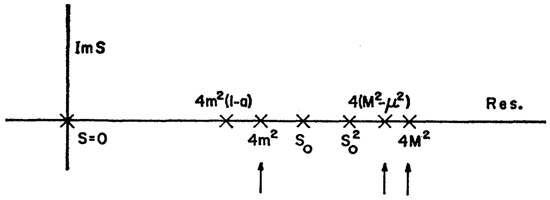

FIG. 5. Singularities in s of partial-wave amplitudes. The singular points, which occur for the normal case also, are indicated by arrows.

of s, for which with s', t' fixed at an endpoint of the integration, the argument of $Q_l(z)$ functions can be $z=+1, -1, -\infty$. In addition, there are singularities coming from the vanishing of denominators $(s'-s)$. We list the singular points of each of the $a_l^{(i)}$ of the Eq. (3.2). $a_l^{(1)}(s)$ is singular at $s=4m^2$, $4(M^2-\mu^2)$, $4M^2$, ∞; $a_l^{(2)}(s)$ at $s=0$, s_0^2, $4M^2$, ∞; $a_l^{(3)}(s)$ at $s=0$, $4(M^2-\mu^2)$, $4m^2(1-a)$, $4m^2$, s_0, $4M^2$, ∞; and $a_l^{(4)}(s)$ at $s=0$, $4(M^2-\mu^2)$, $4m^2(1-a)$, $4m^2$, s_0, $4M^2$, where

$$a=[(M^2-m^2-\mu^2)/(2m\mu)]^2$$
$$s_0=2(M^2+m^2-\mu^2).$$

Notice that the singular points $s=s_0$, $s=s_0^2$, and $s=4(M^2-\mu^2)$ lie between $s=4m^2$ and $s=4M^2$, for a stable Σ particle mass. The singular points of $a_l(s)$ are shown in the Fig. 5.

The reality of the branch points means that the Regge pole parameters have only real singularities in complex s variable. In fact, aside from additional branch points, the Regge behavior of partial waves even with complex singularities is quanitatively very similar to parital waves with normal thresholds. This is very important for calculational purposes, as it allows one to deal with the relatively simple structure of the Regge amplitudes, rather than struggle with complex singularities in the full amplitude. How far this unexpected "tameness" of partial waves extends is difficult to conjecture, if it holds. It is especially important that such behavior exists for the Regge amplitudes, since the full amplitude may be constructed from them, even when a straightforward partial-wave sum may fail to converge.

IV. SHARING THEOREM AND UNITARITY

It follows directly from the extension of partial-wave unitarity to complex l that a singularity in the l plane in one channel, of a set of coupled channels, necessarily appears in all the coupled channels. This shall be referred to as the "sharing theorem." The factoring theorem for the residues of the Regge poles is also a consequence of the same. For a set of coupled amplitudes with only normal thresholds, and obeying Mandelstam representation, the validity of the extended unitarity formula is a consequence of (i) the proper Regge continuation given by Froissart, (ii) the Mandelstam formula for the spectral functions, Eq. (2.2), and (iii) Henrici-Froissart identity for $Q_l(z)$ functions.[4]

The partial-wave unitarity for a set of normal amplitudes reads

$$\mathrm{Im}a_{ij}(l,s)\equiv\frac{1}{2i}[a_{ij}(l,s)-\bar{a}_{ij}(l,s)]$$
$$=\sum_k \rho_k a_{ik}(l,s)\bar{a}_{jk}(l,s)\,, \tag{4.1}$$

where a bar over a function means the complex conjugation of the functional form only, and not of the arguments. Thus, if $f(z)$ is an analytic function of z, so is $\bar{f}(z)$, and is given by $f^*(z^*)$; and

$\rho_k=p_k\theta(s-s_k)/(8\pi\sqrt{s})$ is the phase space factor,

$p_k=$c.m. momentum in the k channel with threshold at $s=s_k$.

As a consequence of (i)–(iii) above, Eq. (4.1) is valid everywhere in the complex l plane; a fact which, as has been pointed out by Squires,[12] is really a consequence of Carlson's theorem, given that the Froissart continuation of $a(l,s)$ from integral values of l allows one to perform a Sommerfeld-Watson transform.

The factoring theorem[13] follows at once from (4.1). Suppose the Regge pole is $l=\alpha(s)$, and that there is no pole at $l=\alpha^*(s)$. One finds by comparing the residues of the pole at $l=\alpha(s)$

$$(\mathrm{Im}\alpha)R=R^\dagger\rho R\,, \tag{4.2}$$

where we have used a matrix notation for the residue matrix $(R)_{ij}=R_{ij}$. From (4.2) it follows, as in the case of Breit-Wigner resonances, that

$$R_{ij}=\gamma_i\gamma_j\,,$$

i.e., R_{ij} is factorizable. There is also a sum rule

$$\sum_k \rho_k|\gamma_k|^2=\mathrm{Im}\alpha\,.$$

The question of whether the sharing theorem holds for amplitudes with complex singularities is an important one physically. The very slow decrease of the nucleus-nucleus cross section with energy means that either there is a cut in the angular momentum plane, causing only a logarithmic decrease for these amplitudes not shared by other communicating channels, or that the cross sections at moderate energies are roughly the asymptotic cross sections. If the latter is true, then, as pointed out by Gell-Mann,[14] the factoring

[12] E. J. Squires, Nuovo Cimento **25**, 242 (1962).
[13] M. Gell-Mann, Phys. Rev. Letters **8**, 263 (1962); V. N. Gribov and I. Ya. Pomeranchuk, *ibid.* **8**, 343 (1962); J. N. Charap and E. J. Squires, Phys. Rev. **127**, 1387 (1962).
[14] M. Gell-Mann, in *Proceedings of the 1962 International Conference on High Energy Physics at CERN* (CERN, Geneva, 1962), p. 541.

theorem and the sharing theorem together cannot be correct since the cross section for nucleus A–nucleus A' scattering goes as $(A^{1/3}+A'^{1/3})^2$ at moderate energies, whereas the sharing-factoring theorem predicts an asymptotic cross section which goes like AA'. On the other hand, if there are cuts present in nucleus-nucleus scattering, we are hard put to explain why they are not demanded by experiment for, say, nucleon-nucleon scattering, if the sharing theorem holds. One thus has to investigate the question of the failure of the sharing theorem for the case with complex singularities. We shall show that this is not the case for the simple example considered here.

In a sense, it is quite apparent that the sharing theorem continues to hold even for amplitudes with complex singularities by the consideration of "the principle of analytic continuation in masses of scattering particles." One writes Eq. (4.1) for those values of masses such that all thresholds are normal. Both sides of Eq. (4.1) are analytic functions of the external squared masses M_i^2; continuation in the M_i^2 cannot thus lead to a violation of the form of (4.1), i.e., extended unitarity. Nevertheless, it is interesting to pursue the question of unitarity in some detail.

In order to explicitly demonstrate extended unitarity, it is very helpful to note that for $s>4m^2$, the expression (3.2) for $a_l(s)$ can be cast into the following more transparent form

$$a(l,s)=\frac{1}{2q^2}\left\{\frac{1}{\pi^2}\int_{4m^2}^{\infty}ds'\int_{t'(s)}^{\infty}dt'\frac{\rho(s',t')}{s'-s}Q_l\left(1+\frac{t'}{2q^2}\right)\right.$$
$$-\frac{2}{\pi}\int_{4(1-a)\mu^2}^{t'(s)}dt'\,\sigma'(s,t')Q_l\left(1+\frac{t'}{2q^2}\right)$$
$$\left.-\frac{2i}{\pi}\int_{4(1-a)\mu^2}^{0}dt'\,\sigma(s,t')Q_l\left(1+\frac{t'}{2q^2}\right)\right\}$$
$$(s>4m^2)\cdots,\quad(4.3)$$

where

$$t'(s)=4\mu^2+\frac{16m^2\mu^2a}{s-4m^2}$$

$$\sigma'(s,t)=\left\{\left(\frac{s}{4m^2}\right)\left(\frac{t}{4m^2}\right)\left[a-\left(\frac{s}{4m^2}-1\right)\left(\frac{t}{4\mu^2}-1\right)\right]\right\}^{-1/2}.$$

The proof of equivalence of this form (4.3), which is valid for only $s>4m^2$, with the expression (3.2) can be given by using Carlson's theorem and noting $a_l(s)$ as given by (3.2) and (4.3) have the same behavior as $|l|\to\infty$, Re$l>N$, and they further coincide for integral values of l. The way we have obtained (4.3) is as follows: We obtain, after very considerable manipulations, that the expression for $A(s,t)$ for $s>4m^2$ can be reduced to

$$A(s,t)=\frac{1}{\pi^2}\int_{4m^2}^{\infty}ds'\int_{t'(s)}^{\infty}dt'\frac{\rho(s',t')}{(s'-s)(t'-t)}$$
$$-\frac{2}{\pi}\int_{4(1-a)\mu^2}^{t'(s)}\frac{dt'\,\sigma'(s,t')}{t'-t}$$
$$-\frac{2i}{\pi}\int_{4(1-a)\mu^2}^{0}\frac{dt'\,\sigma(s,t')}{t'-t}(s>4m^2)\cdots.\quad(4.4)$$

This reduction was already obtained by FMN. In these manipulations, one has to use the explicit form of $\sigma(s,t)$. Then one projects out the continuation (4.3) from (4.4).

Using (4.3), we obtain

$$\mathrm{Im}a(l,s)=\frac{1}{\pi}\int_{z'(s)}^{\infty}dz'\rho(s,z')Q_l(z')$$
$$-\frac{2}{\pi}\int_{z_a(s)}^{1}dz'\sigma(s,z')Q_l(z')\cdots,\quad(4.5)$$

where $t=-2q^2(1-z)$ and $z_a(s)=1+2(1-a)\mu^2/q^2$. Thus, if extended unitarity is satisfied, we must have the right-hand side of (4.5) equal to

$$\sum_k\rho_k a^{(k)}(l,s)\bar{a}^{(k)}(l,s)$$
$$=\sum_k\rho_k\int dz_1dz_2\,A_t^{(k)}(s,z_1)A_t^{(k)}(s,z_2)$$
$$\times Q_l(z_1)Q_l(z_2)\cdots.\quad(4.6)$$

Using (i), the expression given in Sec. II before, for $\rho(s,z)$, which is

$$\rho(s,z)=\sum_k\rho_k\int dz_1dz_2\,A_t^{(k)}(s,z_1)A_t^{*(k)}(s,z_2)$$
$$\times[K(z,z_1,z_2)]^{-1/2}\cdots\quad(4.7)$$

$$K(z,z_1,z_2)=z^2+z_1^2+z_2^2-1-2zz_1z_2\cdots,\quad(4.8)$$

(the integration is over the region over which $K>0$), and (ii) the Henrici-Froissart identity,[4]

$$\int_{z_0}^{\infty}\frac{dz'\,Q_l(z')}{[K(z',z_1,z_2)]^{1/2}}=Q_l(z_1)Q_l(z_2)\quad(4.9)$$
$$z_0=z_1z_2+[z_1^2-1)^{1/2}(z_2^2-1)^{1/2}$$
$$(z_1^2-1)>0,\quad(z_2^2>1),$$

we obtain that the generalized unitarity is satisfied identically for $s>4m^2$ and for $4M^2>s>4m^2$ would be satisfied if

$$\frac{1}{\pi}\int_{z'(s)}^{\infty}dz'\,\sigma(s,z')Q_l(z')-\frac{1}{\pi}\int_{z_a(s)}^{1}dz'\,\sigma(s,z')Q_l(z')$$
$$=\rho_{(\Lambda\Lambda)}\left|\int dz_1\,B_t^{\mathrm{pole}}(s,z_1)Q_l(z_1)\right|^2\cdots,\quad(4.10)$$

i.e.,

$$\int_{z_0}^{\infty}\frac{dz'\,Q_l(z')}{[K(z',z_1,z_2)]^{1/2}}-2\int_{z_a}^{1}\frac{dz'\,Q_l(z')}{[K(z',z_1,z_2)]^{1/2}}=[Q_l(z_1)]^2\cdots\quad(4.11)$$

for

$$z_1^2-1=\frac{4\mu^2}{s-4m^2}+\frac{4(M^2-m^2-\mu^2)^2}{(s-4m^2)(s-4m^2)}<0.$$

That this is so can be seen continuing in z_1 variable the identity (4.9). It should be emphasized that (4.11) is an identity derived by continuation in z_1; the validity of (4.11) does not depend on the assumption of unitarity.

V. CONCLUDING REMARKS

We have so far shown that certain amplitudes with complex singularities do not yield cuts as a direct consequence of the Regge continuation. The problem remains as to what mechanism can produce cuts if they are present. Amati *et al.*[2] have suggested that ordinary elastic unitarity in the crossed channel may yield cuts. But it is obviously insufficient to retain only elastic unitarity in calculating $\mathrm{Im}A(s,t)$. The optical theorem says $\mathrm{Im}A(s,t=0)\sim s\sigma^{\mathrm{total}}$, where σ^{total} is the total cross section, not just elastic cross section σ^{el}. Of course, $\sigma^{el}\sim 1/\ln s$, which simply expresses the shrinkage of the diffraction peak. It is not enough to calculate $\mathrm{Im}A(s,t)$ from elastic unitarity alone; one must include all the *inelastic* channels, which become available at large s. Thus,

$$\mathrm{Im}A(s,t)=\mathrm{Im}A^{(1)}(s,t)+\mathrm{Im}A^{(2)}(s,t),$$

where the first part comes from elastic, and the second from inelastic unitarity. It is perfectly consistent that

$$\mathrm{Im}A^{(1)}(s)\sim s^{\alpha(t)}/\ln s,$$

while $\mathrm{Im}A(s,t)\sim s^{\alpha(t)}$. The inelastic contribution just cancels the apparent cut coming from the elastic contribution. Thus, that argument is inconclusive.

It would be most satisfactory if we could say there are no cuts. However, the situation is inconclusive. If the cuts are present in some channel, they would be present in all channels with which this channel communicates if the sharing theorem continues to hold, and there is no reason to think that it would not. It may well happen that the strength of the branch cut might be greater for processes with prominent anomalous thresholds.

In concluding, we would like to emphasize that Regge amplitudes, even with complex singularities, are remarkably well behaved, and a calculation program based on them would thus probably circumvent to a considerable degree the complexities associated with complex singularities.

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A Mechanism for the Induction of Symmetries Among the Strong Interactions

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A model is constructed in which there are N equally massive vector mesons, which are self-consistent bound states of pairs of these same vector mesons. It is shown that N must be equal to the number of parameters of some compact, semisimple Lie group, and that the renormalized coupling constants must be proportional to the structure constants of the group.

THE strong interactions are known to exhibit isotopic spin symmetry, which is based on the group SU_2. There is also evidence that a further symmetry described by SU_3 exists, although this further symmetry is certainly much more approximate. In this paper, we raise the question of whether these symmetries might have a simple dynamical origin.

A phenomenological symmetry related to a Lie group is understood to mean two things. First, the mass spectrum of particles which have the same spin and parity should show a clustering which can be identified with the multiplet structure corresponding to representations of the group. Second, the S-matrix elements, and in particular, the renormalized coupling constants referring to different particles from the same multiplet should be related, approximately, through Clebsch-Gordan coefficients. It is clear that the origin of such a phenomenological symmetry could be established only through the development of a rather complete understanding of strong-interaction dynamics. Nevertheless,